Volatility Smile Analysis Through the Heston Model

Pablo Marchesi
October 2024

Abstract

This paper presents a simple approach to modeling the volatility smile using the Heston model, which incorporates stochastic volatility. We investigate the impact of different model parameters on the shape of the volatility smile and the resulting probability density function of log stock prices. A comparative analysis is conducted with traditional constant volatility models, such as the Geometric Brownian Motion, highlighting the advantages and limitations of each approach. Our study aims to provide insights into the practical implications of stochastic volatility for option pricing.

1 The Volatility Smile

The Black-Scholes formula revolutionized financial engineering by providing the first closed-form solution for pricing European options. Its introduction sparked significant growth in the global derivatives markets, offering traders and investors a powerful tool for evaluating options and managing risk more effectively.

However, the Black-Scholes model relies on several unrealistic assumptions, such as constant implied volatility and stock prices following a Geometric Brownian Motion (GBM), which assumes log-normal price distribution. The limitations of these assumptions became evident during the 1987 market crash, known as Black Monday, when the Dow Jones index plummeted by 22.6% in a single day. Under the GBM framework, this event would be a 27-standard deviation occurrence—an outcome so improbable that it borders on the impossible. This extreme deviation highlighted the need for more realistic models to capture market behavior.

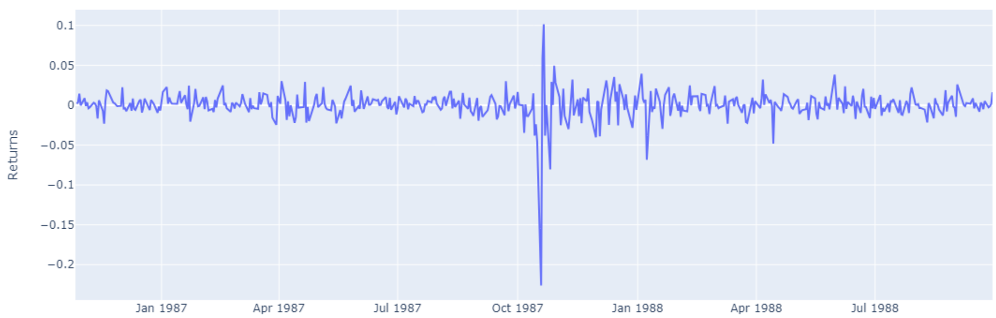

Figure 1: Dow Jones daily returns

Black Monday revealed that large price swings occur far more frequently than predicted by the Geometric Brownian Motion model, leading to fat-tailed distributions that significantly deviate from the log-normal distribution assumed by GBM. Soon after the crash, options traders noticed an unusual pattern: prices for deep out-of-the-money puts were unexpectedly high. In the Black-Scholes framework, higher option prices indicate higher implied volatility. When traders plotted implied volatility against the strike prices of these options, the resulting curve was not flat, as the theory would predict, but exhibited a distinct smile shape for deep out-of-the-money options. This phenomenon, known as the volatility smile, contradicted the assumptions of the Black-Scholes model, where implied volatility should be constant across strike prices.

The need for a more realistic market model became clear, prompting the development of stochastic volatility models. One of the most notable is the Heston model, introduced in 1993. Unlike the Black-Scholes framework, the Heston model accounts for the randomness of volatility itself, producing fat-tailed distributions for stock prices and allowing for the modeling of the volatility smile. This approach offered a significant improvement in capturing market behaviors that the Black-Scholes model could not explain.

2 The Heston Model

The Heston model incorporates stochastic volatility to Geometric Brownian Motion by adding a CIR (Cox, Ingersoll, Ross) process that models the instantaneous variance, ν_t of the GBM. This process is mean-reverting, which implies that volatility tends to return to a long-term mean after large deviations. We can confirm this empirically by looking at the VIX index, which shows the 30-day implied volatility for the S&P500 options.

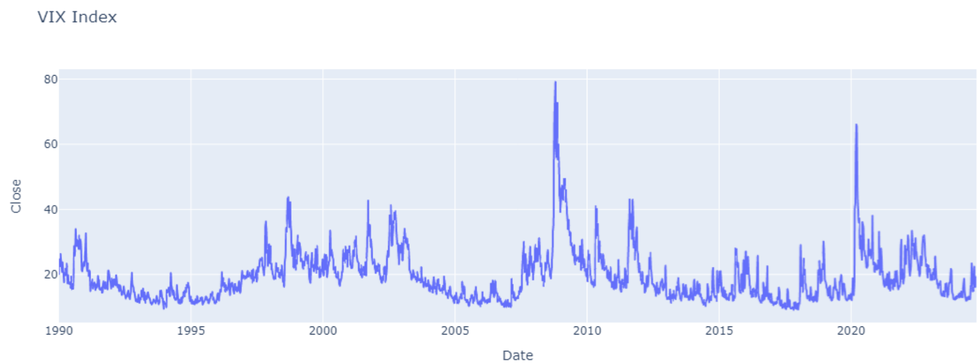

Figure 2: The VIX index presents a mean-reverting behaviour

The dynamics of the Heston model are governed by the following stochastic differential equations:

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S \tag{1}$$
$$d\nu_t = \kappa(\theta - \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^\nu \tag{2}$$
$$\mathbb{E}[dW_t^S \, dW_t^\nu] = \rho \, dt \tag{3}$$

Where:

- S_t is the stock price at time t,
- ν_t is the variance of the asset price at time t,
- μ is the drift rate of the asset price,
- κ is the rate of mean reversion of the variance,
- θ is the long-run mean of the variance,
- ξ is the volatility of volatility,
- ρ is the correlation between the Wiener processes W_t^S and W_t^ν,
- W_t^S and W_t^ν are standard Brownian motions.

We will be focusing on how the main parameters of the Heston model (ν_0, ξ, ρ, κ and θ) affect the volatility smile of put options and the distribution of stock prices. In addition, we will compare these results with the GBM model.

3 The Initial Variance ν_0

The first parameter we will examine is the initial variance, ν_0. If we set all the other parameters to 0, we have a constant volatility model (as GBM). By changing the initial variance, we can modify the width of the PDF. It is also clear that we will have a flat volatility smile, as we are assuming the variance constant.

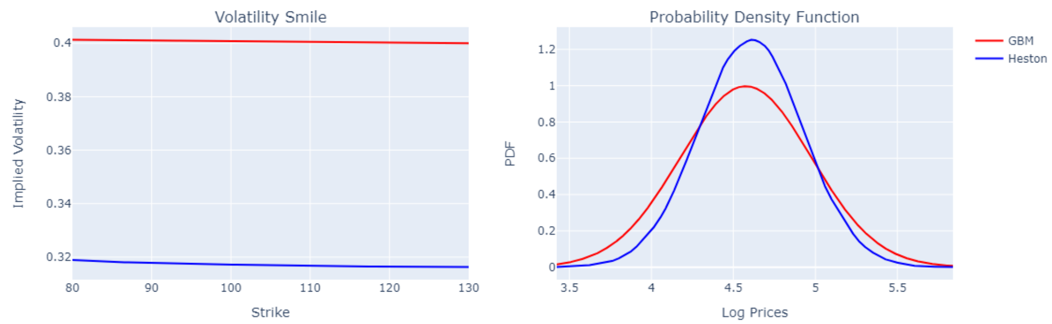

Figure 3: Results for $\nu_0 = 0.1$

4 The Volatility of Volatility ξ

The second parameter of the Heston model is the volatility of volatility (or vol of vol, for short), ξ. This parameter controls the kurtosis of the distribution of log prices. In other words, it controls the tail risk or how fat the tails are. It is also in charge of the curvature of the smile. If set $\xi > 0$ we get:

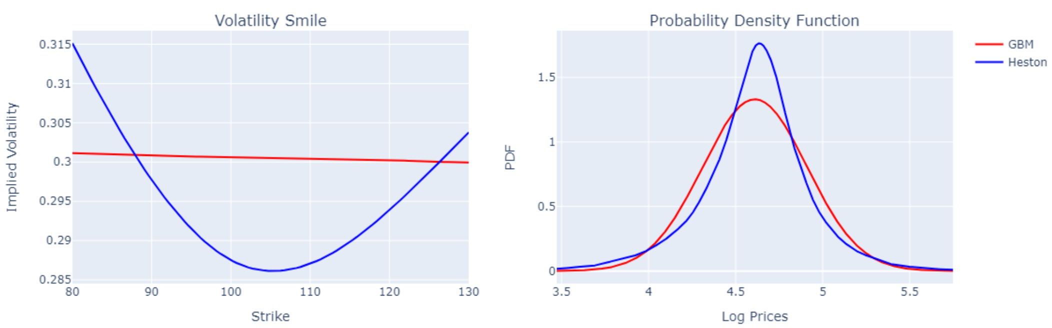

Figure 4: Results for $\xi = 0.5$

As we can see in the image, we now have curved implied volatility for the Heston model, and the distribution of stock prices is considerably more fat-tailed than in the GBM model. This is a more realistic market model that is able to capture extreme events such as Black Monday.

5 The Correlation Coefficient ρ

The stock prices and the volatility are usually negative-correlated. This is due to the fact that whenever a market crash occurs, the volatility spikes significantly. This, again, can be confirmed empirically by taking a look at the VIX index shown before. It is clear that the volatility rises when the market dumps.

In the Heston model, this market behaviour is captured by the third equation of the model, which explains the correlation between stock prices and their volatility. We can adjust this correlation by changing the correlation coefficient, ρ, which we will assume is negative for the above reasons. This parameter controls the asymmetry of the distribution of log prices also known as the skewness.

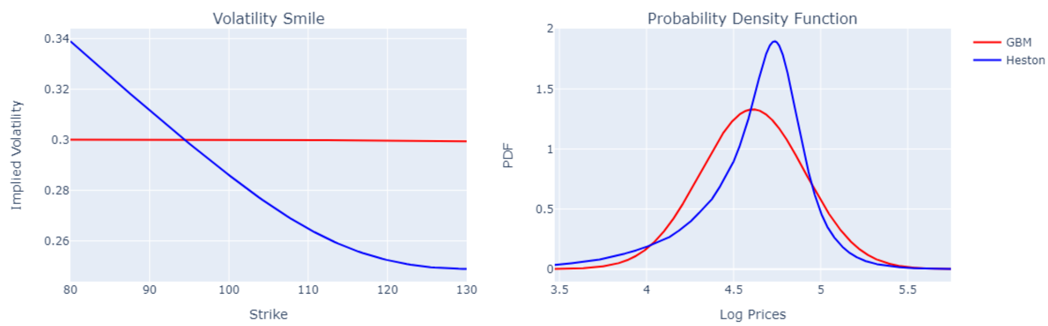

Figure 5: Results for $\rho = -0.5$

By setting a negative correlation coefficient, we can generate negative-skewed PDFs, which assign a higher probability to extreme dumps in the price of stocks. Regarding the volatility smile, we can observe that higher implied volatility is assigned to deep-out-the-money put options. This shows that the investors are willing to pay more for these options because they provide a hedge against the not-so-infrequent market crashes.

6 The Mean-Reversion Parameters κ and θ

Lastly, we have two more parameters, θ, which is in charge of the long-term mean of the volatility, and κ, which controls the speed of the mean-reverting behavior. If we take the expected value of the stochastic volatility process we get:

$$\mathbb{E}[\nu_t] = \nu_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) \tag{4}$$

From this formula, we can deduce that as time t increases, the expected value of the variance will converge to θ when κ is sufficiently large. In other words, with these parameters, we can control the convergence in mean of the volatility process. In the following example, we set θ close to the volatility of the Geometric Brownian Motion, so that both the probability density function and the volatility smile of the Heston model approach those of the GBM when κ is large enough.

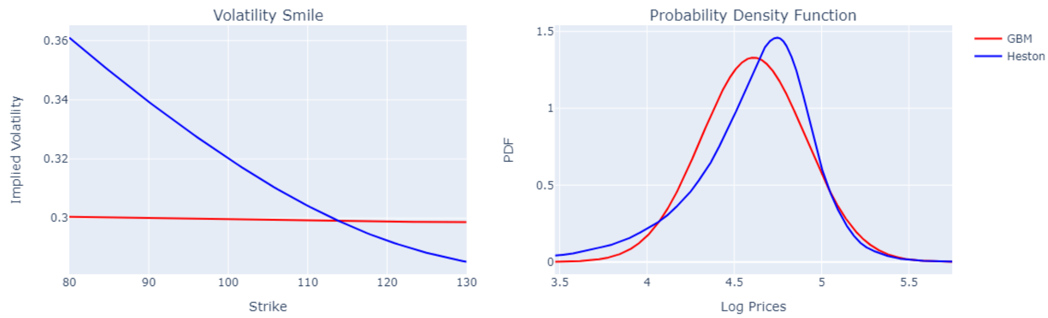

Figure 6: Results for $\kappa = 0.2$ and $\theta = 0.3$

7 Methodology

In this section, we will provide a more detailed explanation of the calculations carried out in this analysis of the volatility smile and the PDFs. The first step involved simulating the Heston and GBM processes using an Euler-Maruyama discretization scheme and then a Monte Carlo simulation. The discretized equations are:

$$S_t = S_{t-1} + \mu S_{t-1}\Delta t + \sqrt{v_{t-1}} S_{t-1}\Delta W_t^S, \tag{5}$$

$$v_t = v_{t-1} + \kappa(\theta - v_{t-1})\Delta t + \xi\sqrt{v_{t-1}}\Delta W_t^v, \tag{6}$$

$$\Delta W_t^\nu = \rho \Delta W_t^S + \sqrt{1-\rho^2}\Delta W_t \tag{7}$$

Where we have computed ΔW_t^S and ΔW_t as:

$$\Delta W_t^S = Z^{(1)}\sqrt{\Delta t} \quad \text{where} \quad Z^{(1)} \sim \mathcal{N}(0,1) \tag{8}$$

$$\Delta W_t = Z^{(2)}\sqrt{\Delta t} \quad \text{where} \quad Z^{(2)} \sim \mathcal{N}(0,1) \tag{9}$$

Equation (7) matches with (3) as:

$$\mathbb{E}[\Delta W_t^S \Delta W_t^\nu] = \mathbb{E}[\Delta W_t^S(\rho\Delta W_t^S + \sqrt{1-\rho^2}\Delta W_t)] = \rho\mathbb{E}[(\Delta W_t^S)^2] = \rho\Delta t$$

For the Geometric Brownian Motion, we have discretized the model as:

$$S_t = S_{t-1} + \mu S_{t-1}\Delta t + \sigma S_{t-1}\Delta W_t^S \tag{10}$$

Note that, in this case, the standard deviation (therefore the variance) is constant and is represented by the parameter σ.

After simulating both processes, we can calculate the value of a put option by discounting the expected value of the payoffs at expiration under a risk-neutral probability measure $\mathbb{Q}$, given a strike price K, the risk-free rate r and the time to maturity T. This calculation can be summarised with the following formula:

$$V_{MC}(S_T, K, T, r) = e^{-rT}\,\mathbb{E}_{\mathbb{Q}}[\max(K - S_T, 0)] \tag{11}$$

Having obtained the option prices, we can calculate the implied volatility by inverting the Black-Scholes formula using the Newton method. This calculation enables us to obtain the implied volatility for a given option price (calculated with the Monte Carlo approach). The price of a European put option, based on the Black-Scholes model, is given by the formula:

$$V_{BS}(S_0, K, T, r, \sigma) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \tag{12}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

To calculate the implied volatility, we use the Newton-Raphson method. The steps are as follows:

- Start with an initial guess for volatility σ_0
- Iteratively update the volatility using the following formula until convergence:

  $$\sigma_{n+1} = \sigma_n - \frac{V_{BS}(\sigma_n) - V_{MC}}{\text{Vega}(\sigma_n)}$$

  where:

  - $V_{BS}(\sigma_n)$ is the option price calculated using the Black-Scholes formula for the current volatility guess σ_n
  - V_{MC} is the option price calculated with the Monte Carlo approach
  - $\text{Vega}(\sigma_n)$ is the sensitivity of the option price with respect to volatility, given by:

    $$\text{Vega}(\sigma) = S_0\Phi'\left(d_1(\sigma)\right)\sqrt{T}$$

  - $d_1(\sigma)$ is defined as:

    $$d_1(\sigma) = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

- The iteration continues until $|V_{BS}(\sigma_n) - V_{MC}| < \text{tol}$ where tol is a predefined tolerance.

Lastly, we have to compute the PDF of both the Heston model and the GBM. In the case of the Heston model, we have used a Kernel Density Estimation (KDE) using as input data the log prices of the simulation S_T.

For a more detailed explanation of the implementation in Python, please click here.

8 Conclusions

After studying the parameters of the Heston model, we have confirmed the advantages of this stochastic volatility approach. It allows us to create a more realistic market model compared to the Geometric Brownian Motion. Specifically, we have generated more accurate stock price distributions, which exhibit fat tails and negative skewness and have successfully captured the dynamics of implied volatility.

References

[1] Sergio de la Riva Polanco. Comprehensive analysis and calibration of the heston model for stochastic volatility in option pricing. *Unpublished*, 2024.

[2] Robin Dunn, Paloma Hauser, Tom Seibold, and Hugh Gong. Estimating option prices with heston's stochastic volatility model. *Valparaiso University*, 2014.

[3] J Gatheral. *The volatility surface: A Practitioner's Guide*. John Wiley and Sons, Inc, 2011.

[4] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343, 1993.

[5] Scott Patterson. *The Quants: The maths geniuses who brought down Wall Street*. Random House, 2012.